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EXPOSED AND DENTING POINTS IN DUALS OF OPERATOR SPACES

BY

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ABSTRACT

We study the extremal structure of the dual unit balls of various operator spaces. Mainly, we show that the classes of [w*-] strongly exposed, [w*-] exposed, and denting points in the dual unit balls of spaces of compact operators between Banach spaces X and Y are completely — and in a canonical way — determined by the corresponding classes of points in the unit balls of the (bi-)duals of the factor spaces X and Y. Applications to the duality of operator spaces and differentiability properties of the norm in operator spaces are given.

0.1. Introduction

Various spaces of compact operators between Banach spaces X and Y can be realized as (linear subspaces of) the space $K_{w^*}(X^*, Y)$ of all compact and weak*-weakly continuous linear operators from X^* into Y, endowed with the usual norm:

(a) $K(X, Y) = K_{w^*}(X^{**}, Y)$

$k \mapsto k^{**}$,

(b) $X \tilde{\otimes}_\varepsilon Y \hookrightarrow K_{w^*}(X^*, Y)$

$x \otimes y \mapsto \{x^* \mapsto (x^*x)y\}$, and

$X \tilde{\otimes}_\varepsilon Y = K_{w^*}(X^*, Y)$ whenever either of X and Y has the approximation property.

(c) $C(K, X) = K_{w^*}(X^*, C(K)) = C(K)\tilde{\otimes}_\varepsilon X$

$F \mapsto \{x^* \mapsto x^*F\}$ (K compact Hausdorff).

(d) $cca(\Sigma, X) = K_{w^*}(X^*, ca(\Sigma)) = ca(\Sigma)\tilde{\otimes}_\varepsilon X$

$\Phi \mapsto \{x^* \mapsto x^*\Phi\}$

(where $cca(\Sigma, X)$ denotes the space of countably additive, compact range

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vector measures from a σ-algebra Σ into X, endowed with the semi-variation norm).

The starting point for our investigation here is the work of Tseitlin [22] and ourselves [18], where it is shown that for any linear subspace H of $K_{w^*}(X^*, Y)$, containing $X \otimes Y$: $X \otimes Y \subset H \subset K_{w^*}(X^*, Y)$, the extreme points in the unit ball B_{H^*} of H^* are completely determined by those in the unit balls B_{X^*} and B_{Y^*} of X^* and Y^* respectively:

$$\text{ext}\, B_{H^*} = \text{ext}\, B_{X^*} \otimes \text{ext}\, B_{Y^*},$$

where $x^* \otimes y^*(h) = (hx^*, y^*)$ for all $h \in H$.

In this paper, we prove analogous results for the classes of [w*-] strongly exposed, [w*-] exposed, and denting points in the dual unit ball of H.

In particular, the isometrical isomorphism of proposition (a) above specializes these results to a description of these classes of extreme points in the dual of the space $K(X, Y)$ of compact linear operators from X into Y.

For the case of w*-strongly exposed points, however, our results apply as well to any linear subspace H of even the spaces $W(X, Y)$ and $L(X, Y)$ of all weakly compact and of all bounded linear operators from X into Y, respectively. In particular, we shall show:

$$\begin{aligned} \text{w}^*\text{-sexp}\, B_{K(X,Y)^*} &= \text{w}^*\text{-sexp}\, B_{W(X,Y)^*} = \text{w}^*\text{-sexp}\, B_{L(X,Y)^*} \\ &= \text{sexp}\, B_X \otimes \text{w}^*\text{-sexp}\, B_{Y^*}. \end{aligned}$$

Among the applications are results on the duality between operator spaces, and — starting from Smulyan's classical characterization of w*-[strongly] exposed points by differentiability properties of the norm — several results on Gateaux- and Fréchet-differentiability of the norm in operator spaces.

Parts of the results have been presented at the Banach Center Seminar on "Geometry of Banach Spaces" in Warszawa in March, 1983, at the 3rd Paderborn Conference on "Functional Analysis" in May, 1983, and at the conference on "Factorization of Linear Operators and the Geometry of Banach Spaces" in Columbia, Missouri, USA, in June, 1984.

0.2. Terminology and notation

Throughout, all linear spaces will be assumed to be over the reals. Basically, we follow the terminology and notation of Dunford and Schwartz [5].

The spaces of bounded, of weakly compact, and of compact linear operators from X into Y — X and Y Banach spaces — are denoted by $L(X, Y)$, $W(X, Y)$, and $K(X, Y)$, respectively. In accordance with our definition of $K_{w^*}(X^*, Y)$, we denote by $L_{w^*}(X^*, Y)$ the space of *all* weak*-weakly continuous linear operators from X^* into Y, endowed with the usual operator norm.

The closed unit ball of a Banach space Z will be denoted by B_Z, the unit sphere by S_Z.

We briefly recall the definitions of [w*-] (strongly) exposed points (for [w*-] denting points, see section 3):

(i) $z_0^* \in S_{Z^*}$ is called an exposed (respectively w*-exposed) point of B_{Z^*}, if there exists $f_0 \in S_{Z^{**}}$ (respectively $f_0 \in S_Z$) such that $1 = f_0(z_0^*) > f_0(z^*)$ for all $z^* \in B_{Z^*} \setminus \{z_0^*\}$. The set of all exposed (respectively w*-exposed) points of B_{Z^*} is denoted by $\exp B_{Z^*}$ (respectively w*-$\exp B_{Z^*}$).

(ii) $z_0^* \in S_{Z^*}$ is called a strongly exposed (respectively w*-strongly exposed) point of B_{Z^*}, if there exists $f_0 \in S_{Z^{**}}$ (respectively $f_0 \in S_Z$) such that $f_0(z_0^*) = 1$, and, given any $\varepsilon > 0$, there exists $\delta > 0$ such that $B_{Z^*} \cap (f_0 > 1 - \delta) \subset B(z_0^*; \varepsilon)$.

Here, $(f_0 > 1 - \delta)$ denotes the set $\{z^* \in Z^* \mid f_0(z^*) > 1 - \delta\}$, and $B(z_0^*; \varepsilon) = \{z^* \in Z^* \mid \|z^* - z_0^*\| \leq \varepsilon\}$.

The set of all strongly exposed (respectively w*-strongly exposed) points of B_{Z^*} will be denoted by $\operatorname{sexp} B_{Z^*}$ (respectively w*-$\operatorname{sexp} B_{Z^*}$).

For equivalent definitions, and a general discussion of these special classes of extreme points, and their importance in the geometry of Banach spaces, the reader is referred to Diestel [3], Diestel and Uhl [4], and to Phelps [14, 15].

1. Strongly exposed points

Classical results by V.L. Smul'yan [20, 21] relate the exposed point structure of B_{Z^*} to differentiability properties of the norm in Z.

SMUL'YAN [20, 21]. *A point $z_0 \in S_z$ w*-exposes [w*-strongly exposes] B_{Z^*} at $z_0^* \in S_{Z^*}$, if and only if the norm of Z is Gateaux-differentiable [Fréchet-differentiable] at z_0 with differential z_0^*.*

Our results on w*-[strongly] exposed points in the dual unit ball of $K_{w^*}(X^*, Y)$ will be based on bilinear versions of Smul'yan's results.

1.1. THEOREM. *Assume that U and V are closed norming linear subspaces of X^* and Y^*, respectively, and that H is a linear subspace of the space $B(X, Y)$ of all continuous bilinear forms on $X \times Y$, containing $U \otimes V$: $U \otimes V \subset H \subset B(X, Y)$.*

Let $B_0 \in S_H$, and $\phi = \| \ \|_{B(X,Y)}$. Then the following propositions are equivalent:

(a) *ϕ is F-differentiable at B_0.*

(b) *There exist* $(x_0, y_0) \in S_X \times S_Y$ *such that*

(i) $B_0(x_0, y_0) = 1$, *and*

(ii) *whenever* $(x_n)_n \subset B_X$ *and* $(y_n)_n \subset B_Y$ *are such that* $(B_0(x_n, y_n))_n$ *tends to* 1, *then there exist* $\alpha \in \{-1, 1\}$ *and subsequences* $(x_{n_i})_i$ *and* $(y_{n_i})_i$ *such that* $x_{n_i} \to \alpha x_0$ *and* $y_{n_i} \to \alpha y_0$.

(c) *There exists* $(x_0, y_0) \in S_X \times S_Y$ *such that* B_0 *strongly exposes* $B_{X \otimes_\pi Y}$ *at* $x_0 \otimes y_0$.

(d) $\phi\,|_H = \|\ \|_H$ *is F-differentiable at* B_0.

For a proof of Theorem 1.1, we shall need the following technical lemma.

1.2. Lemma. *Assume that U and V are closed norming linear subspaces of* X^* *and* Y^*, *respectively. Then we have*:

(a) $B_X \otimes B_Y$ *is closed in* $(U \tilde{\otimes}_\varepsilon V)^*$, *and*

(b) *whenever* $(x_n)_n \subset B_X$ *and* $(y_n)_n \subset B_Y$ *are such that* $(x_n \otimes y_n)_n$ *is Cauchy uniformly over* $B_U \otimes B_V$ *and* $\|x_n\| \geq \alpha$, $\|y_n\| \geq \alpha$ *for some* $\alpha > 0$ *and all* $n \in \mathbf{N}$, *then* $(x_n)_n$ *and* $(y_n)_n$ *have Cauchy subsequences.*

Proof. First we note that, since U and V are norming, the maps

$$X \to U^* \qquad\qquad Y \to V^*$$
$$\text{and}$$
$$x \mapsto F_x : \{u \mapsto ux\} \qquad\qquad y \mapsto F_y : \{v \mapsto vy\}$$

are isometrical embeddings, so that

$$x_n \otimes y_n : U \tilde{\otimes}_\varepsilon V \to \mathbf{R}$$
$$k \mapsto (kx_n, y_n)$$

are continuous linear functionals on $U \tilde{\otimes}_\varepsilon V$.

Next, we note that proposition (a) is a consequence of proposition (b): Let $(x_n)_n \subset B_X$, $(y_n)_n \subset B_Y$, and $T \in (U \tilde{\otimes}_\varepsilon V)^*$ such that $\|x_n \otimes y_n - T\|_{(U \otimes_\varepsilon V)^*} \to 0$.

If $(x_n)_n$ has a subsequence $(x_{n_i})_i$ converging to zero in X, then $(x_{n_i} \otimes y_{n_i})_i$ tends to zero in $(U \tilde{\otimes}_\varepsilon V)^*$, so that $T = 0 \in B_X \otimes B_Y$.

So, we assume that neither $(x_n)_n$ nor $(y_n)_n$ has a subsequence converging to zero. Then we can assume that $\|x_n\| \geq \alpha$, $\|y_n\| \geq \alpha$ for some $\alpha > 0$ and all $n \in \mathbf{N}$. Then, according to proposition (b), there exist $(x_0, y_0) \in B_X \times B_Y$ and subsequences $(x_{n_i})_i$ and $(y_{n_i})_i$ such that $\|x_{n_i} - x_0\| \to 0$ and $\|y_{n_i} - y_0\| \to 0$. This implies that

$$\|x_{n_i} \otimes y_{n_i} - x_0 \otimes y_0\|_{(U \otimes_\varepsilon V)^*} \to 0,$$

so that, since $(x_n \otimes y_n)_n$ is Cauchy in $(U \tilde{\otimes}_\varepsilon V)^*$, $T = x_0 \otimes y_0 \in B_X \otimes B_Y$.

For a proof of proposition (b), assume that $(x_n)_n \subset B_X$ and $(y_n)_n \subset B_Y$ are such that $\|x_n\| \geq \alpha$, $\|y_n\| \geq \alpha$ for some $\alpha > 0$ and all $n \in \mathbf{N}$, and that $(x_n \otimes y_n)_n$ is Cauchy uniformly over $B_U \otimes B_V$.

Assume first that some subsequence of $(x_n)_n$ converges to zero with respect to $\sigma(X, U)$. Then, according to a result of Bessaga and Pelczynski [1] (cf. [13, IV.1, Thm. 6, p. 58]), there exists a subsequence $(x_{n_i})_i$ of $(x_n)_n$ which is a basic sequence. Hence, there exist $(x_i^*)_i \subset X^*$ and $C > 0$ such that $x_i^*(x_{n_j}) = \delta_{ij}$ and $\|x_i^*\| \leq C$. (Recall that $\|x_{n_i}\| \geq \alpha > 0$ for all $i \in \mathbf{N}$.) Since $(x_n \otimes y_n)_n$ is Cauchy uniformly over $B_U \otimes B_V$, there exists, for $\varepsilon = \alpha/8$, an index $i_0 \in \mathbf{N}$ such that

(1) $|(x_{n_i} \otimes y_{n_i} - x_{n_j} \otimes y_{n_j}, u \otimes v)| < \alpha/8$ for all $i, j \geq i_0$ and all $u \in CB_{X^*} \cap U$ and $v \in B_{Y^*} \cap V$.

Let $i, j \geq i_0$, $i \neq j$, be fixed, and choose $y^* \in B_{Y^*} \cap V$ such that $y^* y_{n_i} > \|y_{n_i}\| - \alpha/2$ and $x^* \in CB_{X^*} \cap U$ such that $|(x^* - x_i^*, \{x_{n_i}, x_{n_j}\})| < \alpha/8$ (note that w*-cl$(CB_{X^*} \cap U) = CB_{X^*}$). Then we have:

(2) $x^* x_{n_i} = (x^* - x_i^*)(x_{n_i}) + x_i^*(x_{n_i}) > 1 - \alpha/8$, and

(3) $|x^* x_{n_j}| \leq |(x^* - x_i^*, x_{n_j})| + |x_i^*(x_{n_j})| < \alpha/8$.

Now, $x^* \otimes y^*$ is an element of $(CB_{X^*} \cap U) \otimes (B_{Y^*} \cap V)$, but we have:

(4) $|(x_{n_i} \otimes y_{n_i} - x_{n_j} \otimes y_{n_j}, x^* \otimes y^*)| \geq |(x^* x_{n_i})(y^* y_{n_i})| - |(x^* x_{n_j})(y^* y_{n_j})|$
$\geq (1 - \alpha/8)(\|y_{n_i}\| - \alpha/2) - \alpha/8 \geq \alpha/2(1 - \alpha/8) - \alpha/8 > (1/4)\alpha$,

contradicting (1).

We thus have shown that no subsequence of $(x_n)_n$ converges to zero with respect to $\sigma(X, U)$. Similarly, no subsequence of $(y_n)_n$ converges to zero with respect to $\sigma(Y, V)$. In particular, there exist $x_0^* \in B_U$, $\beta > 0$, and a subsequence $(x_{n_i})_i$ of $(x_n)_n$ such that $x_0^*(x_{n_i}) \to \beta$. By assumption, we have:

$$\sup\{|(x_{n_i} \otimes y_{n_i} - x_{n_j} \otimes y_{n_j}, x^* \otimes y^*)| \mid x^* \in B_U, y^* \in B_V\} \to 0 \quad \text{as } i, j \to \infty.$$

This implies that

$$\|x_0^*(x_{n_i}) y_{n_i} - x_0^*(x_{n_j}) y_{n_j}\| = \sup_{y^* \in B_V} |((x_0^*(x_{n_i}) y_{n_i} - x_0^*(x_{n_j}) y_{n_j}), y^*)| \to 0$$

as $i, j \to \infty$.

Hence, the sequence $(x_0^*(x_{n_i}) y_{n_i})_i$ is Cauchy in Y, so that the sequence $(y_{n_i})_i$ itself is Cauchy in Y, for $x_0^*(x_{n_i}) \to \beta > 0$. In an analogous way, we can deduce also that a certain subsequence of $(x_n)_n$ is Cauchy. This completes the proof of Lemma 1.2.

Proof of Theorem 1.1. Clearly, (a) implies (d). In order to see that (c) implies (a), we need only recall that, for any Banach space Z, we have $\operatorname{sexp} B_Z = \text{w*-}\operatorname{sexp} B_{Z^{**}}$, and then use Smul'yan's result quoted above.

We now show that (d) *implies* (b): Again, according to Smul'yan's result, if

$\phi|_H = \| \quad \|_H$ is F-differentiable at B_0, then there exists $T_0 \in S_{H^*}$ such that B_0 strongly exposes B_{H^*} at T_0. Choose $(x_n)_n \subset S_X$ and $(y_n)_n \subset S_Y$ such that $B_0(x_n, y_n) \to 1$. Then the sequence $(\|x_n \otimes y_n - T_0\|_{H^*})_n$ tends to zero, so that $(x_n \otimes y_n)_n$ is uniformly Cauchy over $B_U \times B_V$. According to Lemma 1.2 (b), and since $(x_n)_n \subset S_X$ and $(y_n)_n \subset S_Y$, there exist $(x_0, y_0) \in S_X \times S_Y$ and subsequences $(x_{n_i})_i$ and $(y_{n_i})_i$ such that $x_{n_i} \to x_0$ and $y_{n_i} \to y_0$. This implies that $\|x_{n_i} \otimes y_{n_i} - x_0 \otimes y_0\|_{H^*} \to 0$, so that $T_0 = x_0 \otimes y_0$ and $B_0(x_0, y_0) = 1$.

Now, let $(x_n)_n \subset B_X$ and $(y_n)_n \subset B_Y$ such that $B_0(x_n, y_n) \to 1$. Then, again, $\|x_n \otimes y_n - x_0 \otimes y_0\|_{H^*} \to 0$. In particular, no subsequence of $(x_n)_n$ or $(y_n)_n$ converges to zero. According to Lemma 1.2 (b), there exist $(x_1, y_1) \in B_X \times B_Y$ and subsequences $(x_{n_i})_i$ and $(y_{n_i})_i$ such that $x_{n_i} \to x_1$ and $y_{n_i} \to y_1$. We conclude that $x_0 \otimes y_0 = x_1 \otimes y_1$ on H, hence on $U \otimes V$. Since U and V are norming linear subspaces of X^* and Y^*, this implies that $x_0 = \alpha x_1$ and $y_0 = \alpha y_1$ for some $\alpha \in \{-1, 1\}$. This shows that (d) implies (b).

Next, we show that (b) *implies* (a): We claim that $x_0 \otimes y_0$ is the F-differential of ϕ at B_0. Assume it is not. Then, for some $\varepsilon > 0$, there exists for each $n \in \mathbf{N}$ an element B_n of $B(X, Y)$ with $0 < \|B_n\| < 1/n$ such that

(1) $|\,\|B_0 + B_n\| - \|B_0\| - x_0 \otimes y_0(B_n)| \geq \varepsilon \|B_n\|$ for all $n \in \mathbf{N}$.

Given $n \in \mathbf{N}$, choose $(x_n, y_n) \in B_X \times B_Y$ such that

(2) $(B_0 + B_n)(x_n, y_n) > \|B_0 + B_n\| - (1/n)\|B_n\|$.

Then we have:

$$\begin{aligned}\|B_0\| \geq B_0(x_n, y_n) &= (B_0 + B_n)(x_n, y_n) - B_n(x_n, y_n) \geq (B_0 + B_n)(x_n, y_n) - \|B_n\| \\ &\geq \|B_0 + B_n\| - (1 + 1/n)\|B_n\| \to \|B_0\| \qquad \text{as } n \to \infty,\end{aligned}$$

so that $B_0(x_n, y_n) \to 1$. By assumption, there exist subsequences $(x_{n_i})_i$ and $(y_{n_i})_i$, and $\alpha \in \{-1, 1\}$ such that $x_{n_i} \to \alpha x_0$ and $y_{n_i} \to \alpha y_0$, so that $\|x_{n_i} \otimes y_{n_i} - x_0 \otimes y_0\|_\pi \to 0$. Denoting by b_{n_i} the continuous linear form on $X \tilde{\otimes}_\pi Y$ corresponding to B_{n_i}, we conclude that

$$\begin{aligned}(3)\quad |B_{n_i}(x_{n_i}, y_{n_i}) - B_{n_i}(x_0, y_0)| &= |b_{n_i}(x_{n_i} \otimes y_{n_i} - x_0 \otimes y_0)| \\ &\leq \|B_{n_i}\| \, \|x_{n_i} \otimes y_{n_i} - x_0 \otimes y_0\|_\pi.\end{aligned}$$

Taking into account that

$$\|B_0 + B_{n_i}\| \geq (B_0 + B_{n_i})(x_0, y_0) = \|B_0\| + B_{n_i}(x_0, y_0),$$

(2) and (3) lead to

$$\begin{aligned}(4)\quad 0 &\leq \|B_0 + B_{n_i}\| - \|B_0\| - B_{n_i}(x_0, y_0) \\ &< (B_0 + B_{n_i})(x_{n_i}, y_{n_i}) + (1/n_i)\|B_{n_i}\| - \|B_0\| - B_{n_i}(x_0, y_0) \\ &\leq (B_0(x_{n_i}, y_{n_i}) - \|B_0\|) + \|B_{n_i}\|(\|x_{n_i} \otimes y_{n_i} - x_0 \otimes y_0\|_\pi + (1/n)) \\ &\leq \|B_{n_i}\|(\|x_{n_i} \otimes y_{n_i} - x_0 \otimes y_0\|_\pi + (1/n)).\end{aligned}$$

But this contradicts (1), for $\| x_{n_i} \otimes y_{n_i} - x_0 \otimes y_0 \|_\pi \to 0$. This completes the proof that (b) implies (a).

So far, we have shown that (a), (b) and (d) are equivalent, and that (c) implies (a). We now show that (a) *and* (b) *imply* (c):

If ϕ is F-differentiable at B_0, then B_0 strongly exposes $B_{B(X,Y)^*}$ at some $T_0 \in B_{B(X,Y)^*}$. According to (b), there exists $(x_0, y_0) \in S_X \times S_Y$ such that $B_0(x_0, y_0) = 1$. Hence, $T_0 = x_0 \otimes y_0$, and B_0 strongly exposes $B_{X \otimes_\pi Y}$ at $x_0 \otimes y_0$.

This completes the proof of Theorem 1.1.

We now use Theorem 1.1 to deduce results on the exposed point structure of dual unit balls of operator spaces and on F-differentiability properties of the norm in operator spaces.

1.3. THEOREM. *Assume that $x_0 \in S_X$ and $y_0 \in S_Y$ strongly expose B_{X^*} and B_{Y^*} at x_0^* and y_0^*, respectively. Then $x_0 \otimes y_0$ strongly exposes $B_{X^* \otimes_\pi Y^*}$ at $x_0^* \otimes y_0^*$.*

PROOF. First, we note that $X \otimes Y \subset B(X^*, Y^*) = (X^* \tilde{\otimes}_\pi Y^*)^*$, and that X and Y are (closed) norming linear subspaces of X^{**} and Y^{**}, respectively. By assumption, $x_0 \otimes y_0(x_0^* \otimes y_0^*) = 1$. Now, let $(x_n^*)_n \subset B_{X^*}$ and $(y_n^*)_n \subset B_{Y^*}$ such that $x_0 \otimes y_0(x_n^* \otimes y_n^*) \to 1$. Then, since x_0 and y_0 are strongly exposing B_{X^*} and B_{Y^*} at x_0^* and y_0^*, respectively, we conclude that, for some $\alpha \in \{-1, 1\}$, $x_n^* \to \alpha x_0^*$ and $y_n^* \to \alpha y_0^*$. Hence, proposition (b) of Theorem 1.1 is fulfilled, so that, by proposition (c) of that same result, $x_0 \otimes y_0$ strongly exposes $B_{X^* \otimes_\pi Y^*}$ at $x_0^* \otimes y_0^*$. This completes the proof.

1.4. COROLLARY. *Let H be a linear subspace of $L(X^*, Y^{**})$, containing $X \otimes Y$: $X \otimes Y \subset H \subset L(X^*, Y^{**}) = B(X^*, Y^*)$, and assume that x_0 and y_0 strongly expose B_{X^*} and B_{Y^*} at x_0^* and y_0^*, respectively. Then $x_0 \otimes y_0$ strongly exposes B_{H^*} at $x_0^* \otimes y_0^*$.*

PROOF. According to Theorem 1.3, $x_0 \otimes y_0$ strongly exposes $B_{X^* \otimes_\pi Y^*}$ at $x_0^* \otimes y_0^*$. Hence, $x_0 \otimes y_0$ strongly exposes $B_{(X^* \tilde{\otimes}_\pi Y^*)^{**}} = B_{(L(X^*, Y^{**}))^*}$ at $x_0^* \otimes y_0^*$. To conclude the proof, it now suffices to use the following trivial observation:

Let E and F be Banach spaces, $T \in L(E, F)$ an isometry (into), and $x_0 \in S_E$. Assume that Tx_0 strongly exposes B_{Y^*} at y_0^*. Then x_0 strongly exposes B_{E^*} at $T^* y_0^*$.

REMARKS. (1) Note that, in Corollary 1.4, H can be taken to be any of the spaces $X \tilde{\otimes}_\varepsilon Y$, $K_{w^*}(X^*, Y)$, $L_{w^*}(X^*, Y)$, $K(X^*, Y)$, $K(X^*, Y^{**})$, and $L(X^*, Y^{**})$.

(2) Replacing X by X^*, we can conclude from Corollary 1.4 that, for any H such that $X^* \otimes Y \subset H \subset L(X^{**}, Y^{**})$, we have:

$$\text{sexp}\, B_X \otimes \text{w}^*\text{-sexp}\, B_{Y^*} \subset \text{w}^*\text{-sexp}\, B_{H^*}.$$

Note that here H can particularly be taken to be any of the spaces $X^* \tilde{\otimes}_\varepsilon Y$, $K(X, Y)$, $L(X, Y)$, $K(X^{**}, Y)$, $L(X^{**}, Y)$, $K(X^{**}, Y^{**})$, and $L(X^{**}, Y^{**})$.

If we restrict ourselves to weak*-weakly continuous linear operators, the consequences of Theorem 1.1 derived so far lead to our main results on weak*-strongly exposed points in duals of operator spaces.

1.5. THEOREM. *Let H be any linear subspace of $L_{w^*}(X^*, Y)$, containing $X \otimes Y$: $X \otimes Y \subset H \subset L_{w^*}(X^*, Y)$. Then we have*:

(a) $\text{w}^*\text{-sexp}\, B_{H^*} = \text{w}^*\text{-sexp}\, B_{X^*} \otimes \text{w}^*\text{-sexp}\, B_{Y^*}$, *and, in particular*:

(b) $\text{w}^*\text{-sexp}\, B_{(X \tilde{\otimes}_\varepsilon Y)^*} = \text{w}^*\text{-sexp}\, B_{(K_{w^*}(X^*, Y))^*} = \text{w}^*\text{-sexp}\, B_{(L_{w^*}(X^*, Y))^*}$
$= \text{w}^*\text{-sexp}\, B_{X^*} \otimes \text{w}^*\text{-sexp}\, B_{Y^*}$.

PROOF. From Corollary 1.4, we know that

$$\text{w}^*\text{-sexp}\, B_{X^*} \otimes \text{w}^*\text{-sexp}\, B_{Y^*} \subset \text{w}^*\text{-sexp}\, B_{H^*}.$$

Conversely, assume that $h_0 \in S_H$ strongly exposes B_{H^*} at $T_0 \in S_{H^*}$. Now, note that $X \otimes Y \subset H \subset L(X^*, Y^{**}) = B(X^*, Y^*)$, and that X and Y are (closed) norming linear subspaces of X^{**} and Y^{**}, respectively. Hence, according to proposition (b) of Theorem 1.1, there exists $(x_0^*, y_0^*) \in S_{X^*} \times S_{Y^*}$ such that

(i) $(h_0 x_0^*, y_0^*) = 1$, and

(ii) whenever $(x_n^*)_n \subset B_{X^*}$ and $(y_n^*)_n \subset B_{Y^*}$ are such that $(h_0 x_n^*, y_n^*) \to 1$, then there exist subsequences $(x_{n_i}^*)_i$ and $(y_{n_i}^*)_i$ and $\alpha \in \{-1, 1\}$ such that $x_{n_i}^* \to \alpha x_0^*$ and $y_{n_i}^* \to \alpha y_0^*$.

We conclude from (i) that $T_0 = x_0^* \otimes y_0^*$, and from (ii) that $h_0 x_0^* \in Y$ and $h_0^* y_0^* \in X$ strongly expose B_{Y^*} and B_{X^*} at y_0^* and x_0^*, respectively. This completes the proof.

1.6. THEOREM. *Assume that H is a linear subspace of $L(X, Y)$, containing $X^* \otimes Y$: $X^* \otimes Y \subset H \subset L(X, Y)$. Then we have*:

(a) $\text{w}^*\text{-sexp}\, B_{H^*} = \text{sexp}\, B_X \otimes \text{w}^*\text{-sexp}\, B_{Y^*}$, *and, in particular*:

(b) $\text{w}^*\text{-sexp}\, B_{(K(X,Y))^*} = \text{w}^*\text{-sexp}\, B_{(W(X,Y))^*} = \text{w}^*\text{-sexp}\, B_{(L(X,Y))^*}$
$= \text{sexp}\, B_X \otimes \text{w}^*\text{-sexp}\, B_{Y^*}$.

The proof of this result is analogous to the foregoing one, where this time we use Theorem 1.1 for the situation $X^* \otimes Y \subset H \subset B(X, Y^*)$.

Theorem 1.1 also allows us to deduce Heinrich's result on the strongly exposed points of the unit ball in a projective tensor product.

1.7. COROLLARY ([8, Thms. 1 and 2]). $\operatorname{sexp} B_{X\otimes_\pi Y} = \operatorname{sexp} B_X \otimes \operatorname{sexp} B_Y$.

PROOF. If $B_0 \in S_{B(X,Y)}$ strongly exposes $B_{X\otimes_\pi Y}$ at z_0, then it strongly exposes $B_{(X\otimes_\pi Y)^{**}} = B_{B(X,Y)^*}$ at z_0, i.e. the norm of $B(X, Y)$ is F-differentiable at B_0 with F-derivative z_0. Hence, according to proposition (c) of Theorem 1.1, B_0 strongly exposes $B_{X\otimes_\pi Y}$ at $x_0 \otimes y_0$ for some $(x_0, y_0) \in S_X \times S_Y$. We conclude that $z_0 = x_0 \otimes y_0$, and that $x_0^* = B_0(\cdot, y_0) \in X^*$ and $y_0^* = B_0(x_0, \cdot) \in Y^*$ strongly expose B_X and B_Y at x_0 and y_0, respectively.

Conversely, assume that x_0^* and y_0^* strongly expose B_X and B_Y (and thus also $B_{X^{**}}$ and $B_{Y^{**}}$) at x_0 and y_0, respectively. Then apply Corollary 1.4 in the situation $X^* \otimes Y^* \subset L(X^{**}, Y^{***})$ to conclude that $x_0^* \otimes y_0^*$ strongly exposes $B_{(X\otimes_\pi Y)^{**}}$ (and thus also $B_{X\otimes_\pi Y}$) at $x_0 \otimes y_0$. This completes the proof.

We now turn to F-differentiability properties of the norm in operator spaces.

1.8. THEOREM. *Assume that H is a linear subspace of $L(X^*, Y^{**})$, containing $X \otimes Y$: $X \otimes Y \subset H \subset L(X^*, Y^{**}) = B(X^*, Y^*)$. Let $(x_0, y_0) \in S_X \times S_Y$. Then the norm of H is F-differentiable at $x_0 \otimes y_0$ if and only if the norms of X and Y are F-differentiable at x_0 and y_0, respectively.*

PROOF. Throughout the proof, we make use of Smul'yan's result quoted at the beginning of this section.

First, assume that x_0 and y_0 strongly expose B_{X^*} and B_{Y^*} at x_0^* and y_0^*, respectively. Then, according to Corollary 1.4, $x_0 \otimes y_0$ strongly exposes B_{H^*} at $x_0^* \otimes y_0^*$, so that the norm of H is F-differentiable at $x_0 \otimes y_0$.

Conversely, assume that the norm of H is F-differentiable at $x_0 \otimes y_0$ with F-derivative T_0. Then, by proposition (b) of Theorem 1.1, there exists $(x_0^*, y_0^*) \in S_{X^*} \times S_{Y^*}$ such that

(i) $x_0^* \otimes y_0^*(x_0 \otimes y_0) = 1$, and

(ii) whenever $(x_n^*)_n \subset B_{X^*}$ and $(y_n^*)_n \subset B_{Y^*}$ are such that $x_n^*(x_0) y_n^*(y_0) \to 1$, then there exist subsequences $(x_{n_i}^*)_i$ and $(y_{n_i}^*)_i$ and $\alpha \in \{-1, 1\}$ such that $x_{n_i}^* \to \alpha x_0^*$ and $y_{n_i}^* \to \alpha y_0^*$.

It is clear from (i) and (ii) that, depending on whether $x_0^*(x_0) = 1$ or $x_0^*(x_0) = -1$, x_0 and y_0 strongly expose B_{X^*} and B_{Y^*} at x_0^* and y_0^* or at $-x_0^*$ and $-y_0^*$, respectively. In any case, the norms of X and Y are F-differentiable at x_0 and at y_0, respectively. This completes the proof.

We note one particular case of Theorem 1.8, with X replaced by X^*.

1.9. COROLLARY. *Assume that H is a linear subspace of $L(X^{**}, Y^{**})$, containing $X^* \otimes Y$: $X^* \otimes Y \subset H \subset L(X^{**}, Y^{**})$. In particular, assume that H is* *any of the spaces* $K(X, Y)$, $L(X, Y)$, $K(X^{**}, Y^{**})$, *or* $L(X^{**}, Y^{**})$. *Furthermore, let* $(x_0^*, y_0) \in S_{X^*} \times S_Y$.

Then the norm of H *is* F*-differentiable at* $x_0^* \otimes y_0$ *if and only if the norms of* X^* *and* Y *are* F*-differentiable at* x_0^* *and* y_0, *respectively.*

The first complete study of differentiability properties of the norm in operator spaces is due to S. Heinrich [9]. His results on characterizing points of F-differentiability of the norms in $X \tilde{\otimes}_\varepsilon Y$, $K(X, Y)$, and $L(X, Y)$ can now as well be deduced from our results given so far. (For the case of G-differentiability, see section 2.) We only note one particular case which extends his result that for $k_0 \in K(X, Y)$ the norm of $K(X, Y)$ is F-differentiable at k_0 if and only if the norm of $L(X, Y)$ is F-differentiable at k_0.

1.10. Corollary (compare [9, Cor. 4.2]). *Let* $k_0 \in S_{K(X,Y)}$. *Then the following propositions are equivalent*:

(a) *The norm of* $K(X, Y)$ *is* F*-differentiable at* k_0.

(b) *The norm of* $L(X, Y)$ *is* F*-differentiable at* k_0.

(c) *The norm of* $L(X^{**}, Y^{**})$ *is* F*-differentiable at* k_0^{**}.

For a proof of this result, we need only apply Theorem 1.1 to the situation $X^* \otimes Y \subset H \subset B(X^{**}, Y^*)$.

We now turn to the characterization of general strongly exposed points in duals of operator spaces. The major difference from the special case of w*-strong exposed points arises from the fact that, in order to achieve a result of the form $\operatorname{sexp} B_{H^*} = \operatorname{sexp} B_{X^*} \otimes \operatorname{sexp} B_{Y^*}$, the operator space H must be such that for given $(x_0^{**}, y_0^{**}) \in X^{**} \times Y^{**}$, a functional $x_0^{**} \otimes y_0^{**}$ must be definable as a continuous linear form on H^* in a natural way, i.e. such that

$$x_0^{**} \otimes y_0^{**}(x^* \otimes y^*) = x_0^{**}(x^*) y_0^{**}(y^*) \qquad \text{for all } (x^*, y^*) \in X^* \times Y^*.$$

Thus, we are limited to linear subspaces of $K_{w^*}(X^*, Y)$.

1.11. Theorem. *Assume that* H *is a linear subspace of* $K_{w^*}(X^*, Y)$, *containing* $X \otimes Y$: $X \otimes Y \subset H \subset K_{w^*}(X^*, Y)$. *Then we have*:

$$\operatorname{sexp} B_{H^*} = \operatorname{sexp} B_{X^*} \otimes \operatorname{sexp} B_{Y^*}.$$

1.12. Corollary. *Let* H *be any linear subspace of* $K(X, Y)$, *containing the finite-rank operators*: $X^* \otimes Y \subset H \subset K(X, Y)$. *Then we have*:

$$\operatorname{sexp} B_{H^*} = \operatorname{sexp} B_{X^{**}} \otimes \operatorname{sexp} B_{Y^*}.$$

The proof of Theorem 1.11 will be divided into several steps, some in the form of technical lemmata which will be needed later on for a characterization of denting points as well.

We start with the easy part:

Step 1. $\operatorname{sexp} B_{H^*} \subset \operatorname{sexp} B_{X^*} \otimes \operatorname{sexp} B_{Y^*}$.

Assume that $T_0 \in S_{H^{**}}$ strongly exposes B_{H^*} at $h_0^* \in S_{H^*}$. Then $h_0^* \in \operatorname{ext} B_{H^*}$, so that, according to [2, Prop. 2.1] and [18, Thm. 1.1], $h_0^* = x_0^* \otimes y_0^*$ for some $(x_0^*, y_0^*) \in \operatorname{ext} B_{X^*} \times \operatorname{ext} B_{Y^*}$. Now, consider the maps

$$\begin{aligned} T_1 : X^* &\to Y^{**} \\ x^* &\mapsto \{y^* \mapsto T_0(x^* \otimes y^*)\} \end{aligned}$$

and

$$\begin{aligned} T_2 : Y^* &\to X^{**} \\ y^* &\mapsto \{x^* \mapsto T_0(x^* \otimes y^*)\}. \end{aligned}$$

The assumptions on T_0 and on $h_0^* = x_0^* \otimes y_0^*$ imply that $T_1 x_0^*$ and $T_2 y_0^*$ strongly expose B_{Y^*} and B_{X^*} at y_0^* and x_0^*, respectively. This completes the first part of the proof.

We are now going to show the reverse implication: Assume that $x_0^{**} \in S_{X^{**}}$ and $y_0^{**} \in S_{Y^{**}}$ strongly expose B_{X^*} and B_{Y^*} at $x_0^* \in S_{X^*}$ and at $y_0^* \in S_{Y^*}$, respectively. We will show that $x_0^{**} \otimes y_0^{**}$ strongly exposes B_{H^*} at $x_0^* \otimes y_0^*$.

Step 2. Action of $x_0^{**} \otimes y_0^{**}$ on H^*.

If $h^* \in H^*$, then $h^*|_{X \otimes_\varepsilon Y}$ is an integral operator from X into Y^*, and thus has a canonical extension $\tilde{h}^* = (h^*|_{X \otimes_\varepsilon Y})^{**}$ to an integral operator from X^{**} into Y^*. The action of $x_0^{**} \otimes y_0^{**}$ on H^* is defined by $x_0^{**} \otimes y_0^{**}(h^*) = (\tilde{h}^* x_0^{**}, y_0^{**})$. Observe that, in this way,

$$x_0^{**} \otimes y_0^{**}(x^* \otimes y^*) = x_0^{**}(x^*) y_0^{**}(y^*) \qquad \text{for all } (x^*, y^*) \in X^* \times Y^*.$$

Step 3. We have to show:

(*) *For any sequence $(h_n^*)_n \subset B_{H^*}$ such that $x_0^{**} \otimes y_0^{**}(h_n^*) \to 1$, we have $\|h_n^* - x_0^* \otimes y_0^*\|_{H^*} \to 0$.*

For a proof of this fact, as well as for the proof of our results on denting points in section 3, we need the following two technical lemmata.

Answering a question of ours, W. Schachermeyer provided a proof of a preliminary version of the first lemma. The proof below is somewhat different.

1.13. Lemma. *Given Banach spaces X and Y, let $x^{**} \in B_{X^{**}}$, $y \in B_Y$, and*

$\nu \in M^+(B_{X^*} \times B_{Y^*})$, $\|\nu\| \le 1$, *and consider the upper semicontinuous upper envelope* $(x^{**})^\wedge$ *of* x^{**} *on* (B_{X^*}, w^*):

$$(x^{**})^\wedge(x^*) = \inf_U \sup_{z^* \in U} x^{**}(z^*),$$

the inf *being taken over all* (B_{X^*}, w^*)*-neighbourhoods of* $x^* \in B_{X^*}$. *Then we have*:

$$\left(x^{**}, \int_{B_{X^*} \times B_{Y^*}} y^*(y)x^* d\nu\right) \le \int_{B_{X^*} \times B_{Y^*}} (x^{**})^\wedge \vee (-x^{**})^\wedge(x^*) d\nu.$$

PROOF. The mapping

$$T: L^1(\nu) \to \mathbf{R}$$
$$g \mapsto (x^{**}, \textstyle\int g x^* d\nu)$$

is a continuous linear functional on $L^1(\nu)$ of norm ≤ 1. Hence, there exists $h \in L^\infty(\nu)$, $\|h\|_\infty \le 1$, such that

$$(1) \qquad (x^{**}, \textstyle\int g x^* d\nu) = \int h g d\nu \qquad \text{for all } g \in L^1(\nu).$$

CLAIM 1. $h \le (x^{**})^\wedge$ ν-a.e.

PROOF. Let $\alpha, \beta \in \mathbf{R}$, $\alpha < \beta$, $E = ((x^{**})^\wedge < \alpha) \times B_{Y^*}$,

$$F = \{(x^*, y^*) \in B_{X^*} \times B_{Y^*} \mid h(x^*, y^*) \ge \beta\},$$

and suppose that $\nu(E \cap F) > 0$. Then there exists V w*-open convex with $\bar{V}^{w^*} \subset E$, such that $\nu(V \cap F) > 0$. For, by regularity of ν, there exists K w*-compact, $K \subset E \cap F$ such that $\nu(K) > 0$. Since K is w*-compact and E is weak*-open, there exist finitely many $k_i \in K$ and w*-open convex w*-neighbourhoods $V(k_i)$ of k_i such that $\bar{V}^{w^*}(k_i) \subset E$ and $K \subset \cup V(k_i)$. Since $\nu(K) > 0$, we have $\nu(V(k_i) \cap F) > 0$ for at least one $i \in \{1, \ldots, n\}$.

Now, let $H = V \cap F$. It is not hard to show that

$$(1/\nu(H)) \int_H x^* d\nu \in \pi_{X^*} \bar{V}^{w^*}, \qquad \text{where } \pi_{X^*}: X^* \times Y^* \to X^*$$
$$(x^*, y^*) \mapsto x^*.$$

We thus have that

$$(1/\nu(H)) \int_H x^* d\nu \in \pi_{X^*} \bar{V}^{w^*} \subset \pi_{X^*} E = ((x^{**})^\wedge < \alpha),$$

and conclude that

$$\alpha > (x^{**})^\wedge\left((1/\nu(H))\int_H x^* d\nu\right) \geq \left(x^{**},(1/\nu(H))\int_H x^* d\nu\right)$$

$$= (1/\nu(H))\int_H h(x^*,y^*)d\nu \geq \beta,$$

which contradicts the choice of α and β. We have thus shown that $h \leq (x^{**})^\wedge$ ν-a.e.

This implies in particular that

$$\text{(2)} \qquad \left(x^{**},\int_{B_{X^*}\times B_{Y^*}} x^* d\nu\right) \leq \int_{B_{X^*}\times B_{Y^*}} (x^{**})^\wedge(x^*)d\nu.$$

Now, let $y\cdot\nu = \mu = \mu^+ - \mu^-$, and assume that $0 < \|\mu^+\|, \|\mu^-\| < 1$. Then we have:

$$\left(x^{**},\int y^*(y)x^* d\nu\right) = \left(x^{**},\int x^* d\mu\right)$$

$$= \|\mu^+\|\left(x^{**},\int x^* d\left(\frac{\mu^+}{\|\mu^+\|}\right)\right) - \|\mu^-\|\left(x^{**},\int x^* d\left(\frac{\mu^-}{\|\mu^-\|}\right)\right)$$

$$\text{by (2)} \leq \|\mu^+\|\int (x^{**})^\wedge d\left(\frac{\mu^+}{\|\mu^+\|}\right) + \|\mu^-\|\int (-x^{**})^\wedge d\left(\frac{\mu^-}{\|\mu^-\|}\right)$$

$$\leq \int (x^{**})^\wedge \vee (-x^{**})^\wedge d|y\cdot\nu| \leq \int (x^{**})^\wedge \vee (-x^{**})^\wedge d\nu.$$

This completes the proof of Lemma 1.13.

1.14. Lemma. *Given Banach spaces X and Y, consider any linear subspace H of $K_{w^*}(X^*, Y)$, containing $X\otimes Y$: $X\otimes Y \subset H \subset K_{w^*}(X^*,Y)$, and let $(x_0^*, y_0^*) \in S_{X^*}\times S_{Y^*}$, $(x_n^{**})_n \subset S_{X^{**}}$, $(y_n^{**})_n \subset S_{Y^{**}}$, and $(h_n^*)_n \subset B_{H^*}$ such that*

(1) $(x_n^{**}\otimes y_n^{**})(h_n^*)\to 1$, and $x_n^{**}(x_0^*)y_n^{**}(y_0^*)\geq 0$ for all $n\in\mathbf{N}$.
Moreover, choose $(\lambda_n)_n \subset M^+(B_{X^*}\times B_{Y^*})$, $\|\lambda_n\|\leq 1$, such that

(2) $h_n^*h = \int_{B_{X^*}\times B_{Y^*}}(hx^*,y^*)d\lambda_n$ for all $h\in H$ and all $n\in\mathbf{N}$, and suppose that, for all $0<\delta<1$:

(3) $\lambda_n((B(x_0^*;\delta)\cup B(-x_0^*;\delta))\times(B(y_0^*,\delta)\cup B(-y_0^*;\delta)))\to 1$ as $n\to\infty$.
Then we have: $\|h_n^* - x_0^*\otimes y_0^*\|_{H^*}\to 0$.

Proof. It is clearly enough to show that, for any $0<\delta<1/18$, there exists a subsequence $(h_{n_k}^*)_k$ of $(h_n^*)_n$ such that

$$\|h_{n_k}^* - x_0^*\otimes y_0^*\|_{H^*} < 18\delta \qquad \text{for all } k\in\mathbf{N} \text{ large enough.}$$

In order to simplify the notation, we denote the sets $B(x_0^*;\delta)$, $B(-x_0^*;\delta)$, $B(y_0^*;\delta)$ and $B(-y_0^*;\delta)$ by E_1, E_2, F_1 and F_2, respectively.

We then observe that, according to (3), there exist $\alpha_i \geq 0$, $i \in \{1,\ldots,4\}$, with $\alpha_1+\alpha_2+\alpha_3+\alpha_4=1$, and a subsequence $(\lambda_{n_k})_k$ of $(\lambda_n)_n$ such that

$$\begin{aligned} &\lambda_{n_k}(E_1\times F_1)\to\alpha_1, \qquad \lambda_{n_k}(E_2\times F_1)\to\alpha_2,\\ &\lambda_{n_k}(E_1\times F_2)\to\alpha_3, \qquad \lambda_{n_k}(E_2\times F_2)\to\alpha_4. \end{aligned} \tag{4}$$

Now, for any $h \in B_H$, we have:

$$\begin{aligned}
&\Big|\int (hx^*,y^*)d\lambda_{n_k} - \alpha_1(hx_0^*,y_0^*) - \alpha_2(h(-x_0^*),y_0^*) - \alpha_3(hx_0^*,-y_0^*) \\
&\qquad\qquad - \alpha_4(h(-x_0^*),-y_0^*)\Big| \\
&\leq \int_{(B_{X^*}\setminus E_1\cup E_2)\times(B_{Y^*}\setminus F_1\cup F_2)} |(hx^*,y^*)|d\lambda_{n_k} \\
&\quad + \int_{E_1\times F_1} |(h(x^*-x_0^*),y^*)+(hx_0^*,y^*-y_0^*)|\,d\lambda_{n_k} \\
&\quad + |(hx_0^*,y_0^*)(\lambda_{n_k}(E_1\times F_1)-\alpha_1)| \\
&\quad + \int_{E_2\times F_1} |(h(x^*+x_0^*),y^*)+(h(-x_0^*),y^*-y_0^*)|\,d\lambda_{n_k} \\
&\qquad\qquad + |h(-x_0^*,y_0^*)(\lambda_{n_k}(E_2\times F_1)-\alpha_2)| \\
&\quad + \int_{E_1\times F_2} |(h(x^*-x_0^*),y^*)+(hx_0^*,y^*+y_0^*)|\,d\lambda_{n_k} \\
&\qquad\qquad + |(hx_0^*,-y_0^*)(\lambda_{n_k}(E_1\times F_2)-\alpha_3)| \\
&\quad + \int_{E_2\times F_2} |(h(x^*+x_0^*,y^*)+(h(-x_0^*),y^*+y_0^*)|\,d\lambda_{n_k} \\
&\qquad\qquad + |(hx_0^*,y_0^*)(\lambda_{n_k}(E_2\times F_2)-\alpha_4)| \\
&\leq \lambda_{n_k}((B_{X^*}\setminus E_1\cup E_2)\times(B_{Y^*}\setminus F_1\cup F_2)) + 8\delta + |\lambda_{n_k}(E_1\times F_1)-\alpha_1| \\
&\quad + |\lambda_{n_k}(E_2\times F_1)-\alpha_2| + |\lambda_{n_k}(E_1\times F_2)-\alpha_3| + |\lambda_{n_k}(E_2\times F_2)-\alpha_4|.
\end{aligned}$$

Together with (3) and (4), this implies that

$$\| h_{n_k}^* - (\alpha_1-\alpha_2-\alpha_3+\alpha_4)x_0^*\otimes y_0^*\|_{H^*} < 9\delta \qquad \text{for all } k\in\mathbf{N} \text{ large enough.} \tag{5}$$

Since $\|x_0^*\otimes y_0^*\|=1$, and $\|h_{n_k}^*\|_{H^*}\to 1$, this implies that $|1-|\alpha_1-\alpha_2-\alpha_3+\alpha_4||\leq 9\delta$ as well, so that, in case $(\alpha_1-\alpha_2-\alpha_3+\alpha_4)\geq 0$, we can finally conclude that $\|h_{n_k}^* - x_0^*\otimes y_0^*\|_{H^*} < 18\delta$ for all $k\in\mathbf{N}$ large enough. However, if we assume that $\alpha_1-\alpha_2-\alpha_3+\alpha_4<0$, then we conclude from (5) and (1) that, for $k\in\mathbf{N}$ large enough:

$$\begin{aligned} 9\delta &> \| h^*_{n_k} - (\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4) x^*_0 \otimes y^*_0 \| \\ &\geq x^{**}_{n_k} \otimes y^{**}_{n_k} (h^*_{n_k} + |\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4| x^*_0 \otimes y^*_0) \\ &\geq x^{**}_{n_k} \otimes y^{**}_{n_k} (h^*_{n_k}) \to 1 \qquad \text{as } k \to \infty, \end{aligned}$$

which contradicts the fact that $18\delta < 1$. This completes the proof of Lemma 1.14.

Step 4. We now complete the proof of Theorem 1.11 by proving proposition (*) from Step 3 above:

Let $(h^*_n)_n \subset B_{H^*}$ such that $x^{**}_0 \otimes y^{**}_0(h^*_n) \to 1$, and, according to [7, Ch. I, §4.1, Prop. 18.2], choose $(\lambda_n)_n \subset M'(B_{X^*} \times B_{Y^*})$ with $\|\lambda_n\| = \|h^*_n\|$, and such that

(1) $$h^*_n h = \int (hx^*, y^*) d\lambda_n \qquad \text{for all } h \in H \text{ and all } n \in \mathbf{N}.$$

CLAIM. For all $0 < \delta < 1$, we have

$$\lambda_n((B(x^*_0;\delta) \cup B(-x^*_0;\delta)) \times B_{Y^*}) \to 1 \qquad \text{as } n \to \infty.$$

PROOF. Let $y \in B_Y$, and choose a net $(x_\gamma)_\gamma \subset B_X$ such that $x_\gamma \to x^{**}_0$ with respect to the Mackey topology $\tau(X^{**}, X^*)$ of the dual pair (X^{**}, X^*). From (1), we have that

$$\begin{array}{ccc} (h^*_n x_\gamma, y) = \displaystyle\int_{B_{X^*} \times B_{Y^*}} y^*(y) x^*(x_\gamma) d\lambda_n & = & \left(x_\gamma, \displaystyle\int_{B_{X^*} \times B_{Y^*}} y^*(y) x^* d\lambda_n\right) \\ \downarrow & & \downarrow \\ (\tilde{h}^*_n x^{**}_0, y) & & \left(x^{**}_0, \displaystyle\int y^*(y) x^* d\lambda_n\right) \end{array}$$

We thus conclude from Lemma 1.13 that, for all $y \in B_Y$:

$$(\tilde{h}^*_n x^{**}_0, y) \leq \int_{B_{X^*} \times B_{Y^*}} (x^{**}_0)^\wedge \vee (-x^{**}_0)^\wedge d\lambda_n.$$

Choosing now a net $(y_\beta)_\beta \subset B_Y$ with $y_\beta \to y^{**}_0$ with respect to $\tau(Y^{**}, Y^*)$, we conclude that

(2) $$x^{**}_0 \otimes y^{**}_0(h^*_n) = (\tilde{h}^*_n x^{**}_0, y^{**}_0) \leq \int_{B_{X^*} \times B_{Y^*}} (x^{**}_0)^\wedge \vee (-x^{**}_0)^\wedge d\lambda_n$$

for all $n \in \mathbf{N}$, so that

(3) $$\int_{B_{X^*} \times B_{Y^*}} (x^{**}_0)^\wedge \vee (-x^{**}_0)^\wedge d\lambda_n \to 1 \qquad \text{as } n \to \infty.$$

According to our assumptions on x^*_0 and x^{**}_0, there exists $0 < \eta < 1$ such that

$$(x_0^{**} \geq 1-\eta) \cap B_{X^*} \subset B(x_0^*;\delta) \quad \text{and} \quad (-x_0^{**} \geq 1-\eta) \cap B_{X^*} \subset B(-x_0^*;\delta).$$

Hence, we have $x_0^{**} < 1-\eta$ on the weak*-open set $B_{X^*} \backslash B(x_0^*;\delta)$ and $-x_0^{**} < 1-\eta$ on the weak*-open set $B_{X^*} \backslash B(-x_0^*;\delta)$, so that, according to the definition of the upper envelopes (Lemma 1.13), we conclude that

$$(4) \qquad (x_0^{**})^\wedge \vee (-x_0^{**})^\wedge \leq 1-\eta \qquad \text{on } B_{X^*} \backslash (B(x_0^*;\delta) \cup B(-x_0^*;\delta)).$$

Together with (3), this implies that

$$\begin{aligned} 1 \leftarrow \int (x_0^{**})^\wedge \vee (-x_0^{**})^\wedge d\lambda_n &\leq (1-\eta)\lambda_n((B_{X^*} \backslash B(x_0^*;\delta) \cup B(-x_0^*;\delta)) \times B_{Y^*}) \\ &\quad + \lambda_n((B(x_0^*;\delta) \cup B(-x_0^*;\delta)) \times B_{Y^*}) \\ &\leq (1-\eta) + \eta\lambda_n((B(x_0^*;\delta) \cup B(-x_0^*;\delta)) \times B_{Y^*}) \leq 1. \end{aligned}$$

This implies that $\lambda_n((B(x_0^*;\delta) \cup B(-x_0^*;\delta)) \times B_{Y^*}) \rightarrow 1$ as $n \rightarrow \infty$, which proves our claim.

Analogous arguments show that

$$\lambda_n(B_{X^*} \times (B(y_0^*;\delta) \cup B(-y_0^*;\delta))) \rightarrow 1 \qquad \text{as } n \rightarrow \infty$$

as well, so that

$$(5) \quad \lambda_n((B(x_0^*;\delta) \cup B(-x_0^*;\delta)) \times (B(y_0^*;\delta) \cup B(-y_0^*;\delta)) \rightarrow 1 \qquad \text{as } n \rightarrow \infty.$$

From Lemma 1.14 we finally conclude that $\| h_n^* - x_0^* \otimes y_0^* \|_{H^*} \rightarrow 0$. This completes the proof of Theorem 1.11.

In concluding this section, we use the results given so far to derive a result of Feder and Saphar [6] on the conjugacy of $K(X, Y)$.

1.15. Theorem ([6, Thm. 2]). *Assume that X and Y are reflexive, and that H is a linear subspace of $K(X, Y)$, containing the finite-rank operators: $X^* \otimes Y \subset H \subset K(X, Y)$.*

Then, if H is (isometric to) a dual space, it is reflexive.

Proof. Assume that $H = Z^*$. According to the assumptions, and by our result [18, Thm. 1.9], $Z^{**} = H^*$ has the Radon–Nikodym property. Hence, we have:

$$\begin{aligned} B_{Z^{**}} &= \text{norm cl co}(\text{sexp } B_{Z^{**}}) = \text{norm cl co}(\text{sexp } B_{H^*}) \\ &= \text{norm cl co}(\text{sexp } B_{X^{**}} \otimes \text{sexp } B_{Y^*}) \\ &= \text{norm cl co}(\text{w}^*\text{-sexp } B_{X^{**}} \otimes \text{w}^*\text{-sexp } B_{Y^*}) \end{aligned}$$

$$= \text{norm cl co}(\text{w}^*\text{-sexp } B_{H^*}) = \text{norm cl co}(\text{w}^*\text{-sexp } B_{Z^{**}})$$

$$= \text{norm cl co}(\text{sexp } B_Z) = B_Z,$$

so that Z, and thus H, is reflexive.

To our knowledge, the problem of whether the isomorphic version of this result holds, is open. Compare [16, p. 1006].

2. Exposed points

As in the case of w*-strongly exposed points, we base our investigation of w*-exposed points in the dual of $K_{w^*}(X^*, Y)$ on a non-linear variant of Smul'yan's characterization.

2.1. PROPOSITION. *Let H be a linear subspace of $C(B_{X^*} \times B_{Y^*})$ (B_{X^*} and B_{Y^*} with their respective weak* topologies), containing $X \otimes Y$: $X \otimes Y \subset H \subset C(B_{X^*} \times B_{Y^*})$, and let $h_0 \in S_H$.*

Then the following propositions are equivalent:

(a) *The norm of H is G-differentiable at h_0.*

(b) *There exist $(x_0^*, y_0^*) \in B_{X^*} \times B_{Y^*}$ and $\alpha \in \{-1, 1\}$ such that $h_0(x_0^*, y_0^*) = \alpha$, and, whenever $(x_n^*, y_n^*)_n \subset B_{X^*} \times B_{Y^*}$ and $\beta_n \in \{-1, 1\}$ are such that $\beta_n h_0(x_n^*, y_n^*) \to 1$, then $(\beta_n x_n^* \otimes y_n^*)_n$ clusters at $\alpha x_0^* \otimes y_0^*$ weak* in H^*.*

PROOF. Assume that the norm of H is G-differentiable at h_0 with differential $T_0 \in S_{H^*}$. Choose $(x_0^*, y_0^*) \in B_{X^*} \times B_{Y^*}$ and $\alpha \in \{-1, 1\}$ such that $h_0(x_0^*, y_0^*) = \alpha$, and let $(x_n^*, y_n^*)_n \subset B_{X^*} \times B_{Y^*}$ and $\beta_n \in \{-1, 1\}$ such that $\beta_n h_0(x_n^*, y_n^*) \to 1$. Then $T_0 = \alpha x_0^* \otimes y_0^*$, and $\beta_n x_n^* \otimes y_n^* \to \alpha x_0^* \otimes y_0^*$ weak* in H^*, for h_0 w*-exposes B_{H^*} at T_0.

Conversely, assume that proposition (b) is fulfilled, but that $\alpha x_0^* \otimes y_0^*$ is not a G-differential of the norm of H at h_0. Then there exist $h \in S_H$, $\varepsilon > 0$, and, for any $n \in \mathbf{N}$, scalars $\lambda_n \in \mathbf{R}$, $0 < \lambda_n < 1/n$, such that

$$(1) \qquad |\|h_0 + \lambda_n h\| - \|h_0\| - \lambda_n \alpha h(x_0^*, y_0^*)| \geq \varepsilon |\lambda_n| \qquad \text{for all } n \in \mathbf{N}.$$

Choose $(x_n^*, y_n^*)_n \subset B_{X^*} \times B_{Y^*}$ and $(\beta_n)_n \subset \{-1, 1\}$ such that

$$(2) \qquad \beta_n (h_0 + \lambda_n h)(x_n^*, y_n^*) = \|h_0 + \lambda_n h\| \qquad \text{for all } n \in \mathbf{N}.$$

Then we have:

$$1 = \|h_0\| \geq \beta_n h_0(x_n^*, y_n^*) = \beta_n (h_0 + \lambda_n h)(x_n^*, y_n^*) - \beta_n h(x_n^*, y_n^*)\lambda_n$$

$$\geq \|h_0 + \lambda_n h\| - |\lambda_n| \|h\| \to \|h_0\| = 1 \qquad \text{as } n \to \infty,$$

so that, according to the assumption:

(3) $(\beta_n x_n^* \otimes y_n^*)_n$ clusters at $\alpha x_0^* \otimes y_0^*$ weak* in H^*.

From the inequality

$$\|h_0 + \lambda_n h\| \geq \alpha(h_0 + \lambda_n h)(x_0^*, y_0^*) = \|h_0\| + \lambda_n \alpha h(x_0^*, y_0^*),$$

we conclude, by using (2), that

$$\begin{aligned}\varepsilon|\lambda_n| &\leq \|h_0 + \lambda_n h\| - \|h_0\| - \lambda_n \alpha h(x_0^*, y_0^*)\\ &= \beta_n(h_0 + \lambda_n h)(x_n^*, y_n^*) - \|h_0\| - \lambda_n \alpha h(x_0^*, y_0^*) \qquad (4)\\ &\leq |\lambda_n|\,|\beta_n h(x_n^*, y_n^*) - \alpha h(x_0^*, y_0^*)| \qquad \text{for all } n \in \mathbf{N}.\end{aligned}$$

This contradicts (3), and the proof of Proposition 2.1 is thus complete.

We next use Proposition 2.1 to single out the technical part of the proof of our results on w*-exposed points in duals of operator spaces.

2.2. Proposition. *Let H be a linear subspace of $C(B_{X^*} \times B_{Y^*})$, containing $X \otimes Y$: $X \otimes Y \subset H \subset C(B_{X^*} \times B_{Y^*})$, and assume that*

$$-\delta_{(-x^*, y^*)}\big|_H = -\delta_{(x^*, -y^*)}\big|_H = \delta_{(x^*, y^*)}\big|_H \qquad \textit{for any pair } (x^*, y^*) \in B_{X^*} \times B_{Y^*}.$$

Moreover, let $(x_0, y_0) \in S_X \times S_Y$ and $(x_0^*, y_0^*) \in S_{X^*} \times S_{Y^*}$. Then we have: If x_0 and y_0 w-expose B_{X^*} and B_{Y^*} at x_0^* and y_0^*, respectively, then $x_0 \otimes y_0$ w*-exposes B_{H^*} at $x_0^* \otimes y_0^*$.*

(Compare J.A. Johnson [11].)

Proof. We use criterion (b) of Proposition 2.1. Assume that $(x_n^*, y_n^*)_n \subset B_{X^*} \times B_{Y^*}$ and $(\beta_n)_n \subset \{-1, 1\}$ are such that $\beta_n x_n^*(x_0) y_n^*(y_0) \to 1$. Then there exist subsequences $(\beta_{n_i} x_{n_i}^*(x_0))_i$ and $(y_{n_i}^*(y_0))_i$, and $\alpha \in \{-1, 1\}$, such that $\alpha \beta_{n_i} x_{n_i}^*(x_0) \to 1$ and $\alpha y_{n_i}^*(y_0) \to 1$, so that by assumption, $\beta_{n_i} x_{n_i}^* \to \alpha x_0^*$ and $y_{n_i}^* \to \alpha y_0^*$ weak* in X^* and Y^*, respectively. We conclude that, for any $h \in H$:

$$\beta_{n_i} h(x_{n_i}^*, y_{n_i}^*) = h(\beta_{n_i} x_{n_i}^*, y_{n_i}^*) \to h(\alpha x_0^*, \alpha y_0^*) = h(x_0^*, y_0^*),$$

which, according to criterion (b) of Proposition 2.1, shows that $x_0 \otimes y_0$ w*-exposes B_{H^*} at $x_0^* \otimes y_0^*$.

2.3. Theorem. *Let H be any linear subspace of $K_{w^*}(X^*, Y)$, containing $X \otimes Y: X \otimes Y \subset H \subset K_{w^*}(X^*, Y)$. Then we have:*

$$\text{w*-exp } B_{H^*} = \text{w*-exp } B_{X^*} \otimes \text{w*-exp } B_{Y^*}.$$

2.4. COROLLARY. *Let H be any linear subspace of $K(X, Y)$, containing the finite-rank operators. Then we have*:

$$\text{w}^*\text{-exp}\, B_{H^*} = \text{w}^*\text{-exp}\, B_{X^{**}} \otimes \text{w}^*\text{-exp}\, B_{Y^*}.$$

PROOF OF THEOREM 2.3. Whenever $x_0^* \in S_{X^*}$ and $y_0^* \in S_{Y^*}$ are w*-exposed by $x_0 \in S_X$ and $y_0 \in S_Y$, then, according to Proposition 2.2, $x_0^* \otimes y_0^*$ is w*-exposed by $x_0 \otimes y_0$.

Conversely, assume that $h_0 \in S_H$ w*-exposes B_{H^*} at $T_0 \in S_{H^*}$. Then T_0 is an extreme point of B_{H^*}, and thus is of the form $T_0 = x_0^* \otimes y_0^*$ with $(x_0^*, y_0^*) \in \text{ext}\, B_{X^*} \times \text{ext}\, B_{Y^*}$, cf. [2, Prop. 2.1] and [18, Thm. 1.1]. It is now easy to check that $x_0 = h_0(\cdot, y_0^*) \in X$ and $y_0 = h_0(x_0^*, \cdot) \in Y$ w*-expose B_{X^*} and B_{Y^*} at x_0^* and y_0^*, respectively. This completes the proof.

We also note the G-differentiability counterpart of Theorem 1.8 of section 1.

2.5. COROLLARY. *Let H be any linear subspace of $C_{\text{odd}}(B_{X^*} \times B_{Y^*})$, containing $X \otimes Y$: $X \otimes Y \subset H \subset C_{\text{odd}}(B_{X^*} \times B_{Y^*})$, where*

$$C_{\text{odd}}(B_{X^*} \times B_{Y^*}) = \{f \in C(B_{X^*} \times B_{Y^*}) \mid -f(-x^*, y^*) = -f(x^*, -y^*) = f(x^*, y^*)$$
$$\textit{for all } (x^*, y^*) \in B_{X^*} \times B_{Y^*}\}.$$

Moreover, let $(x_0, y_0) \in S_X \times S_Y$.

Then the norm of H is G-differentiable at $x_0 \otimes y_0$ if and only if the norms of X and Y are G-differentiable at x_0 and y_0, respectively.

PROOF. Again, the sufficiency part of the statement follows from Proposition 2.2. Assume now that the norm of H is G-differentiable at $h_0 = x_0 \otimes y_0$. Then, according to Theorem 2.3, there exists $(x_0^*, y_0^*) \in S_{X^*} \times S_{Y^*}$ such that $x_0^*(x_0) y_0^*(y_0) = 1$, so that $\alpha x_0^*(x_0) = 1$ and $\alpha y_0^*(y_0) = 1$ for some $\alpha \in \{-1, 1\}$. Assume now that, for some $x_1^* \in B_{X^*}$, $x_1^*(x_0) = 1$. Then $x_1^*(x_0)(\alpha y_0^*)(y_0) = 1$, so that, by assumption on $x_0 \otimes y_0$, $\alpha x_1^* \otimes y_0^* = x_0^* \otimes y_0^*$ on H. This implies that $x_1^*(x) \alpha y_0^*(y) = x_0^*(x) y_0^*(y)$ for all $(x, y) \in X \times Y$, and thus that $x_1^* = \alpha x_0^*$. Hence, x_0 w*-exposes B_{X^*} at αx_0^*. The analogous argument in the Y-factor completes the proof.

Among the consequences of our discussion of w*-exposed points is an extension of the following characterization of smooth points in $K(l_2)$ by J.R. Holub:

J.R. HOLUB [10, Thm. 3.3]. *An operator $T \in K(l_2)$ with $\| T \| = 1$ is a smooth point of $K(l_2)$ if and only if, whenever $\| Tx_1 \| = \| Tx_2 \| = 1$ for some $x_1, x_2 \in S_{l_2}$, then $x_1 = \pm x_2$.*

2.6. PROPOSITION. *Assume that Y is smooth, and let $h_0 \in K_{w^*}(X^*, Y)$ with $\|h_0\| = 1$. Then the norm of $K_{w^*}(X^*, Y)$ is G-differentiable at h_0 if and only if, whenever $\|h_0 x_1^*\| = \|h_0 x_2^*\| = 1$ for some $x_1^*, x_2^* \in S_{X^*}$, then $x_1^* = \pm x_2^*$.*

2.7. COROLLARY. *Assume that Y is smooth, and let $k_0 \in S_{K(X,Y)}$. Then the norm of $K(X, Y)$ is G-differentiable at k_0 if and only if, whenever $\|k_0^{**} x_1^{**}\| = \|k_0^{**} x_2^{**}\| = 1$ for some $x_1^{**}, x_2^{**} \in S_{X^{**}}$, then $x_1^{**} = \pm x_2^{**}$.*

PROOF OF PROPOSITION 2.6. Assume that h_0 is a smooth point of $B_{K_{w^*}(X^*,Y)}$, and that $\|h_0 x_1^*\| = \|h_0 x_2^*\| = 1$ for some $x_1^*, x_2^* \in S_{X^*}$. Then there exist $y_1^*, y_2^* \in S_{Y^*}$ such that $(h_0 x_1^*, y_1^*) = (h_0 x_2^*, y_2^*)$. This implies that $x_1^* \otimes y_1^* = x_2^* \otimes y_2^*$ on $K_{w^*}(X^*, Y)$, and thus on $X \otimes Y$, so that $x_1^* = \pm x_2^*$.

For the converse, we first note that there exists $(x_0^*, y_0^*) \in S_{X^*} \times S_{Y^*}$ such that $(h_0 x_0^*, y_0^*) = 1$, and then use criterion (b) of Proposition 2.1. Let $(x_n^*, y_n^*)_n \subset B_{X^*} \times B_{Y^*}$ and $(\beta_n)_n \subset \{-1, 1\}$ such that $h_0(\beta_n x_n^*, y_n^*) = \beta_n h_0(x_n^*, y_n^*) \to 1$. Then $(\beta_n x_n^*)_n$ and $(y_n^*)_n$ cluster weak* in X^* and Y^* at $x_1^* \in B_{X^*}$ and $y_1^* \in B_{Y^*}$, respectively. Hence, $(\beta_n x_n^* \otimes y_n^*)_n$ clusters weak* in $B_{(K_{w^*}(X^*,Y))^*}$ at $x_1^* \otimes y_1^*$. Since $\|h_0 x_0^*\| = \|h_0 x_1^*\| = 1$, we conclude from our assumption that $x_1^* = \alpha x_0^*$ for some $\alpha \in \{-1, 1\}$. But we also have $(h_0 x_0^*, y_0^*) = 1 = (h_0 x_1^*, y_1^*) = (h_0 x_0^*, \alpha y_1^*)$, so that, since Y is smooth, $y_0^* = \alpha y_1^*$. This implies that $x_1^* \otimes y_1^* = x_0^* \otimes y_0^*$ on $K_{w^*}(X^*, Y)$, and, since $(\beta_n x_n^* \otimes y_n^*)_n$ clusters weak* in H^* at $x_1^* \otimes y_1^*$, an appeal to criterion (b) of Proposition 2.1 completes the proof.

We finally note that, as in the case of Fréchet-differentiability (section 1), our discussion so far also includes Heinrich's results [9] on G-differentiability of the norms in $X \tilde{\otimes}_\varepsilon Y$ and in $K(X, Y)$.

We now determine the form of general exposed points in the dual of $K_{w^*}(X^*, Y)$.

2.8. THEOREM. *Let H be any linear subspace of $K_{w^*}(X^*, Y)$, containing $X \otimes Y$: $X \otimes Y \subset H \subset K_{w^*}(X^*, Y)$. Then we have:*

$$\exp B_{H^*} = \exp B_{X^*} \otimes \exp B_{Y^*}.$$

2.9. COROLLARY. *Let H be any linear subspace of $K(X, Y)$, containing the finite-rank operators. Then we have:*

$$\exp B_{H^*} = \exp B_{X^{**}} \otimes \exp B_{Y^*}.$$

PROOF OF THEOREM 2.8. We start with the easy part: If $T_0 \in S_{H^{**}}$ exposes B_{H^*} at $h_0^* \in S_{H^*}$, then, since $h_0^* \in \operatorname{ext} B_{H^*}$, h_0^* is of the form $h_0^* = x_0^* \otimes y_0^*$ with $(x_0^*, y_0^*) \in \operatorname{ext} B_{X^*} \times \operatorname{ext} B_{Y^*}$. It is then easy to see that $x_0^{**} \in X^{**}$, defined by

$x_0^{**}(x^*) = T_0(x^* \otimes y_0^*)$, and $y_0^{**} \in Y^{**}$, defined by $y_0^{**}(y^*) = T_0(x_0^* \otimes y^*)$, expose B_{X^*} and B_{Y^*} at x_0^* and y_0^*, respectively.

For the converse, we assume now that $x_0^{**} \in S_{X^{**}}$ and $y_0^{**} \in S_{Y^{**}}$ expose B_{X^*} and B_{Y^*} at x_0^* and y_0^*, respectively. Furthermore, we assume that, for some $h_0^* \in S_{H^*}$, we have

$$(1) \qquad x_0^{**} \otimes y_0^{**}(h_0^*) = 1.$$

We have to show that $h_0^* = x_0^* \otimes y_0^*$.

According to Proposition 1.4 of our paper [18], it suffices to show that

$$T = h_0^* \big|_{X \otimes Y} = \delta_{(x_0^*, y_0^*)} \big|_{X \otimes Y},$$

for x_0^* and y_0^* are extreme points of B_{X^*} and B_{Y^*}, respectively.

Step 1. (a) In the course of the proof, we shall make use of the following trivial fact:

Any $f \in C(B_{X^*} \times B_{Y^*})$ is w_X^*-w_Y^*-uniformly continuous, and, consequently, is $\sigma(X^{***}, X^{**})\big|_{B_{X^*}} \times \sigma(Y^{***}, Y^{**})\big|_{B_{Y^*}}$-uniformly continuous as well. Thus, f has a unique extension to a function $\tilde{f} \in C((B_{X^{***}}, w^*) \times (B_{Y^{***}}, w^*))$ of the same norm. Considering the corresponding isometrical embedding

$$C(B_{X^*} \times B_{Y^*}) \to C(B_{X^{***}} \times B_{Y^{***}})$$
$$f \mapsto \tilde{f}$$

and its adjoint, we conclude that

(2) for any $\mu \in M(B_{X^{***}} \times B_{Y^{***}})$, there exists $\nu_\mu \in M(B_{X^*} \times B_{Y^*})$ such that $\int f d\nu_\mu = \int \tilde{f} d\mu$ for all $f \in C(B_{X^*} \times B_{Y^*})$.

(b) Recall from Step 2 in the proof of Theorem 1.11 (section 1) the action of $x_0^{**} \otimes y_0^{**}$ on H^*:

$$x_0^{**} \otimes y_0^{**}(h^*) = ((h^* \big|_{X \tilde{\otimes}_\varepsilon Y})^{**} x_0^{**}, y_0^{**}) \qquad \text{for all } h^* \in H^*.$$

According to [7, Ch. I, §4.1, Prop. 18.2], there exist $\nu_T \in M_1^+(B_{X^*} \times B_{Y^*})$ and $\mu_T \in M_1^+(B_{X^{***}} \times B_{Y^{***}})$ such that $T = h_0^* \big|_{X \tilde{\otimes}_\varepsilon Y} \in (X \tilde{\otimes}_\varepsilon Y)^*$ and $T^{**} \in (X^{**} \tilde{\otimes}_\varepsilon Y^{**})^*$ have the representations

$$(3) \qquad Th = \int_{B_{X^*} \times B_{Y^*}} (hx^*, y^*) d\nu_T \qquad \text{for all } h \in X \tilde{\otimes}_\varepsilon Y$$

and

$$(4) \qquad T^{**}h = \int_{B_{X^{***}} \times B_{Y^{***}}} (hx^{***}, y^{***}) d\mu_T \qquad \text{for all } h \in X^{**} \tilde{\otimes}_\varepsilon Y^{**}.$$

Let $(x, y) \in X \times Y$, and denote by F_x and F_y their respective images in X^{**} and Y^{**} under the canonical embeddings. Then we have:

$$\int_{B_{X^*} \times B_{Y^*}} x \otimes y \, d\nu_T = (Tx, y) = (T^{**}F_x, F_y) = \int_{B_{X^{***}} \times B_{Y^{***}}} F_x \otimes F_y \, d\mu_T$$

$$= \int_{B_{X^*} \times B_{Y^*}} x \otimes y \, d\nu_{\mu_T},$$

where ν_{μ_T} is associated with μ_T according to fact (a) above. We conclude that

$$T = \nu_{\mu_T} \big|_{X \tilde{\otimes}_\varepsilon Y}. \tag{5}$$

Step 2. Recall that we have to show that

$$T = h_0^* \big|_{X \otimes Y} = \delta_{(x_0^*, y_0^*)} \big|_{X \otimes Y}.$$

Using (5), we accomplish this by showing that $\operatorname{supp} \nu_{\mu_T} \subset \{(\pm x_0^*, \pm y_0^*)\}$.

From the equality $1 = x_0^{**} \otimes y_0^{**}(h_0^*) = (T^{**}x_0^{**}, y_0^{**})$, and the assumptions on (x_0^*, y_0^*) and (x_0^{**}, y_0^{**}), we conclude that

$$T^{**}x_0^{**} = y_0^* \quad \text{and} \quad T^* y_0^{**} = x_0^*. \tag{6}$$

Thus, for any $y \in Y$, we have

$$(y_0^*, y) = (T^{**}x_0^{**}, y) = \int_{B_{X^{***}} \times B_{Y^{***}}} x_0^{**} \otimes F_y \, d\mu_T \leq \int_{B_{X^{***}} \times B_{Y^{***}}} 1 \otimes |F_y| \, d\mu_T$$

$$= \int_{B_{X^*} \times B_{Y^*}} 1 \otimes |y| \, d\nu_{\mu_T} = \int_{B_{Y^*}} |y| \, d\nu_{\mu_T} \big|_{C(B_{Y^*})}.$$

According to Lemma 1.5 in our paper [18], this implies that $\operatorname{supp} \nu_{\mu_T} \big|_{C(B_{Y^*})} \subset \{\pm y_0^*\}$.

Starting from (4) with $T_0^* y_0^{**} = x_0^*$, we derive in an analogous way that $\operatorname{supp} \nu_{\mu_T} \big|_{C(B_{X^*})} \subset \{\pm x_0^*\}$.

We have thus shown that $\operatorname{supp} \nu_{\mu_T} \subset \{(\pm x_0^*, \pm y_0^*)\}$, so that

$$h_0^* \big|_{X \otimes Y} = T = \nu_{\mu_T} \big|_{X \otimes Y} = \delta_{(x_0^*, y_0^*)} \big|_{X \otimes Y}.$$

This completes the proof of Theorem 2.8.

3. Denting points

We first recall the definitions of [w*-] denting points.

3.1. Definition (cf. [14]). A point $z_0^* \in S_{Z^*}$ is called a *denting point* (respectively: w*-*denting point*) *of* B_{Z^*} if, for every $\varepsilon > 0$, $z_0^* \notin \| \ \|$-clco$(B_{Z^*} \backslash B(z_0^*;\varepsilon))$ (respectively: $z_0^* \notin \sigma(Z^*, Z)$-clco$(B_{Z^*} \backslash B(z_0^*;\varepsilon))$).

Equivalently (using separation theorems), $z_0^* \in S_{Z^*}$ is a denting point (respectively w*-denting point) of B_{Z^*} if, for every $\varepsilon > 0$, there exist $0 < \alpha \leq \varepsilon$ and $f_\varepsilon \in S_{Z^{**}}$ (respectively $f_\varepsilon \in S_Z$) such that

(i) $f_\varepsilon(z_0^*) > 1 - \alpha$, and

(ii) $B_{Z^*} \cap (f_\varepsilon > 1 - \alpha) \subset B(z_0^*;\varepsilon)$.

The set of [w*-] denting points of B_{Z^*} will be denoted by [w*-] dent B_{Z^*}.

For a general discussion of denting points and their relevance in geometric aspects of the Radon–Nikodym property, we again refer to [3,4] and [14,15].

In our paper [19], we investigated the form of w*-denting points in duals of operator spaces. We showed that, for any linear subspace H of $L_{w^*}(X^*, Y)$, containing $X \otimes Y$, we have:

$$\text{w*-dent } B_{H^*} = \text{w*-dent } B_{X^*} \otimes \text{w*-dent } B_{Y^*} \qquad \text{([19,Thm.3]).}$$

As a particular consequence, we derived the following result ([19,Thm.4]):

$$\begin{aligned}\text{w*-dent } B_{K(X,Y)^*} &= \text{w*-dent } B_{W(X,Y)^*} = \text{w*-dent } B_{L(X,Y)^*} \\ &= \text{dent } B_X \otimes \text{w*-dent } B_{Y^*}.\end{aligned}$$

In this section, we take up the problem of characterizing the form of general denting points in duals of operator spaces. Besides the techniques of [19], this requires the techniques developed in lemmata 1.13 and 1.14 of section 1.

3.2. Theorem. *Let H be a linear subspace of $K_{w^*}(X^*, Y)$, containing $X \otimes Y$: $X \otimes Y \subset H \subset K_{w^*}(X^*, Y)$. Then we have*:

$$\text{dent } B_{H^*} = \text{dent } B_{X^*} \otimes \text{dent } B_{Y^*}.$$

3.3. Corollary. *Let H be a linear subspace of $K(X, Y)$, containing the finite-rank operators. Then we have*:

$$\text{dent } B_{H^*} = \text{dent } B_{X^{**}} \otimes \text{dent } B_{Y^*}.$$

Proof of Theorem 3.2. Again, we start with the easy part: Assume that $T_0 \in \text{dent } B_{H^*}$. Then $T_0 \in \text{ext } B_{H^*}$, and thus $T_0 = x_0^* \otimes y_0^*$ with $(x_0^*, y_0^*) \in \text{ext } B_{X^*} \times \text{ext } B_{Y^*}$, by [2, Prop. 2.1] and [18, Thm. 1.1]. Let $\varepsilon > 0$; then there exist $0 < \alpha \leq \varepsilon$ and $S \in S_{H^{**}}$ such that

(i) $S(x_0^* \otimes y_0^*) > 1 - \alpha$, and

(ii) $B_{H^*} \cap (S > 1 - \alpha) \subset B(x_0^* \otimes y_0^*;\varepsilon)$.

Consider the map

$$k: Y^* \to X^{**}$$
$$y^* \mapsto \{x^* \mapsto S(x^* \otimes y_0^*)\},$$

and let $x^{**} = ky_0^*$. Then $0 < \gamma = \|x^{**}\| \le 1$. We deduce for $x_\varepsilon^{**} = (1/\gamma)x^{**}$ that

$$1 \ge x_\varepsilon^{**}(x_0^*) = (1/\gamma)S(x_0^* \otimes y_0^*) > (1/\gamma)(1-\alpha),$$

and, if $x^* \in B_{X^*} \cap (x^{**} > (1/\gamma)(1-\alpha))$, then

$$1-\alpha = (1/\gamma)(1-\alpha)\gamma < \|x^{**}\| x_\varepsilon^{**}(x^*) = k(y_0^*)(x^*) = S(x^* \otimes y_0^*),$$

so that, by (ii):

$$\|x^* - x_0^*\| \le \|x^* \otimes y_0^* - x_0^* \otimes y_0^*\|_{H^*} \le \varepsilon.$$

This, and an analogous argument for y_0^*, shows that $x_0^* \in \operatorname{dent} B_{X^*}$ and $y_0^* \in \operatorname{dent} B_{Y^*}$, and completes this part of the proof.

We now proceed to show that, conversely, $\operatorname{dent} B_{X^*} \otimes \operatorname{dent} B_{Y^*} \subset \operatorname{dent} B_{H^*}$: Let $(x_0^*, y_0^*) \in \operatorname{dent} B_{X^*} \times \operatorname{dent} B_{Y^*}$. Then we have:

(1) There exists sequences $0 < \alpha_i < \beta_i < 1$, $\alpha_i \to 1$, and $0 < \varepsilon_i \to 0$, and $(x_i^{**})_i \subset S_{X^{**}}$ such that $x_i^{**}(x_0^*) > \beta_i$, and $B_{X^*} \cap (x_i^{**} > \alpha_i) \subset B(x_0^*; \varepsilon_i)$, and accordingly, sequences $0 < \alpha_i' < \beta_i' < 1$, $\alpha_i' \to 1$, and $0 < \varepsilon_i' \to 0$, and $(y_i^{**})_i \subset S_{Y^{**}}$ such that $y_i^{**}(y_0^*) > \beta_i'$, and $B_{Y^*} \cap (y_i^{**} > \alpha_i') \subset B(y_0^*; \varepsilon_i')$.

We conclude that for all $n \in \mathbf{N}$:

$$x_0^* \otimes y_0^* \notin \| \quad \|_{H^*}\text{-clco}\left(\bigcup_{i,j \le n} B_{H^*} \cap (x_i^{**} \otimes y_j^{**} \le \beta_i \beta_j') \right), \tag{2}$$

using our result [19, Prop. 1] that $x_0^* \otimes y_0^*$ is a w*-denting point, and thus an extreme point, of $B_{(X^{**} \otimes_\varepsilon Y^{**})^*}$. Hence, there exist sequences $(h_n^{**})_n \subset S_{H^{**}}$, and $0 < \rho_n < 1$ such that

$$h_n^{**}(x_0^* \otimes y_0^*) > \rho_n > \sup_{h^* \in S_n} h_n^{**}(h^*),$$

where

$$S_n = \bigcup_{i,j \le n} B_{H^*} \cap (x_i^{**} \otimes y_j^{**} \le \beta_i \beta_j').$$

Let $K_n = B_{H^*} \cap (h_n^{**} > \rho_n)$, and choose $h_n^* \in K_n$ for all $n \in \mathbf{N}$. We have to show

$\|h_n^* - x_0^* \otimes y_0^*\|_{H^*} \to 0$. Again, we first choose $(\lambda_n)_n \subset M^+(B_{X^*} \times B_{Y^*})$ with $\|\lambda_n\| = \|h_n^*\|$, and such that

$$(3) \qquad h_n^* h = \int_{B_{X^*} \times B_{Y^*}} (hx^*, y^*) d\lambda_n \qquad \text{for all } h \in H \text{ and all } n \in \mathbf{N}.$$

CLAIM. $\lambda_n((B(x_0^*;\delta) \cup B(-x_0^*;\delta)) \times B_{Y^*}) \to 1$ as $n \to \infty$ for all $0 < \delta < 1$.

PROOF. Suppose that there exist $0 < \delta < 1$, $0 < \beta < 1$, and a sequence $(n_l)_l \subset \mathbf{N}$ such that $\lambda_{n_l}((Bx_0^*;\delta) \cup B(-x_0^*;\delta)) \times B_{Y^*}) \le \beta$ for all $l \in \mathbf{N}$. It is straightforward to construct strictly increasing sequences $(i_k)_k$, $(j_k)_k$, $(l_k)_k \subset \mathbf{N}$, and $(\alpha_{i_k})_k$, $(\beta_{i_k})_k$, $(\beta'_{j_k})_k$, and $(n_{l_k})_k \subset \mathbf{N}$ such that

$$(4) \qquad \min\{i_{k+1}, j_{k+1}\} > n_{l_k} > \max\{n_{l_{k-1}}, i_k, j_k\} \quad \text{and} \quad \alpha_{i_k} < \beta_{i_k} \beta'_{j_k} \qquad \text{for all } k \in \mathbf{N}.$$

This implies that $\alpha_{i_k} < \beta_{i_k}\beta'_{j_k} < (\tilde{h}^*_{n_{l_k}} x^{**}_{i_k}, y^{**}_{j_k})$ for all $k \in \mathbf{N}$. Relabeling, we can thus start from the following situation:

$$(5) \qquad \lambda_n((B(x_0^*;\delta) \cup B(-x_0^*;\delta)) \times B_{Y^*}) \le \beta \quad \text{and} \quad \alpha_n < (\tilde{h}_n^* x_n^{**}, y_n^{**}) \qquad \text{for all } n \in \mathbf{N}.$$

Choose $(x_\gamma)_\gamma \subset B_X$ with $x_\gamma \to x_n^{**}$ with respect to $\tau(X^{**}, X^*)$, and $y \in B_Y$. Then we have, by (3):

$$\begin{array}{ccc} h_n^*(x_\gamma \otimes y) = \int x_\gamma \otimes y d\lambda_n & = & (x_\gamma, \int y^*(y) x^* d\lambda_n) \\ \downarrow & & \downarrow \\ (\tilde{h}_n^* x_n^{**}, y) & & (x_n^{**}, \int y^*(y) x^* d\lambda_n). \end{array}$$

Now, choose $(y_\xi)_\xi \subset B_Y$ with $y_\xi \to y_n^{**}$ with respect to $\tau(Y^{**}, Y^*)$. Then $(\tilde{h}_n^* x_n^{**}, y_\xi) \to (\tilde{h}_n^* x_n^{**}, y_n^{**}) > \alpha_n$. This shows that, for all $n \in \mathbf{N}$, there exists $y_n \in B_Y$ such that

$$(6) \qquad \alpha_n < (\tilde{h}_n^* x_n^{**}, y_n) = (x_n^{**}, \int y^*(y_n) x^* d\lambda_n) \qquad \text{for all } n \in \mathbf{N}.$$

Thus, if we let $x_n^* = \int y^*(y_n) x^* d\lambda_n$, then, by (1):

$$(7) \qquad \|x_n^* - x_0^*\| \to 0.$$

Let $E_\delta = (B(x_0^*;\delta) \cup B(-x_0^*;\delta)) \times B_{Y^*})$, and E_δ^c its complement in $B_{X^*} \times B_{Y^*}$, and

$$a_n^* = (1/\lambda_n(E_\delta)) \int_{E_\delta} y^*(y_n) x^* d\lambda_n \quad \text{and} \quad b_n^* = (1/\lambda_n(E_\delta^c)) \int_{E_\delta^c} y^*(y_n) x^* d\lambda_n.$$

Then $x_n^* = \lambda_n(E_\delta) a_n^* + \lambda_n(E_\delta^c) b_n^*$ for all $n \in \mathbf{N}$. We observe that $(a_n^*)_n$ clusters

$\sigma(X^{***},X^{**})$ at some $a_0^{***}\in B_{X^{***}}$, $(b_n^*)_n$ clusters $\sigma(Y^{***},Y^{**})$ at some $b_0^{***}\in B_{Y^{***}}$, $(\lambda_n(E_\delta))_n$ clusters at some $0\le r\le\beta<1$, and $(\lambda_n(E_\delta^c))_n$ at some $0\le s$ with the following properties:

$$J_{X^*}x_0^*=ra_0^{***}+sb_0^{***},\quad\text{and}\quad r+s=1,\quad 0\le r<1 \tag{8}$$

(where J_{X^*} is the natural embedding of X^* into X^{***}). We conclude that $J_{X^*}x_0^*=b_0^{***}$, for $J_{X^*}x_0^*\in\text{w}^*\text{-dent}\,B_{X^{***}}\subset\text{ext}\,B_{X^{***}}$.

At this point, we choose $i_0\in\mathbf{N}$ such that $B_{X^*}\cap(x_{i_0}^{**}>\alpha_{i_0})\subset B(x_0^*;\delta)$, and thus $B_{X^*}\cap(-x_{i_0}^{**}>\alpha_{i_0})\subset B(-x_0^*;\delta)$, too. Then we have: $(J_{X^*}x_0^*)(x_{i_0}^{**})=x_{i_0}^{**}(x_0^*)>\beta_{i_0}$, and

$$\begin{aligned}\beta_{i_0}<(J_{X^*}x_0^*)(x_{i_0}^{**})&=b_0^{***}(x_{i_0}^{**})\\&=\lim_\gamma x_{i_0}^{**}(b_{n_\gamma}^*)\quad\text{for some subnet }(b_{n_\gamma}^*)_\gamma\text{ of }(b_n^*)_n.\end{aligned} \tag{9}$$

But, for any γ, we have:

$$\begin{aligned}(x_{i_0}^{**},b_{n_\gamma}^*)&=\left(x_{i_0}^{**},(1/\lambda_{n_\gamma}(E_\delta^c))\int_{E_\delta^c}y^*(y_{n_\gamma})x^*d\lambda_{n_\gamma}\right)\\&=(x_{i_0}^{**},\textstyle\int y^*(y_{n_\gamma})x^*d((1/\lambda_{n_\gamma}(E_\delta^c))\chi_{E_\delta^c}\lambda_{n_\gamma}))\\&\le(1/\lambda_{n_\gamma}(E_\delta^c))\int_{E_\delta^c}(x_{i_0}^{**})^\wedge\vee(-x_{i_0}^{**})^\wedge(x^*)d\lambda_{n_\gamma},\end{aligned} \tag{10}$$

where the last inequality is a consequence of Lemma 1.13. However, $(x_{i_0}^{**})^\wedge\vee(-x_{i_0}^{**})^\wedge\le\alpha_{i_0}$ on E_δ^c, for $x_{i_0}^{**}\le\alpha_{i_0}$ and $-x_{i_0}^{**}\le\alpha_{i_0}$ on the weak*-open set $B_{X^*}\backslash B(x_0^*;\delta)\cup B(-x_0^*;\delta)$. We thus conclude from (9) and (10) that $\alpha_{i_0}<\beta_{i_0}<b_0^{***}(x_{i_0}^{**})\le\alpha_{i_0}$, which is a contradiction.

This proves that $\lambda_n((B(x_0^*;\delta)\cup B(-x_0^*;\delta))\times B_{Y^*})\to1$ as $n\to\infty$. By analogous arguments, it can be shown that

$$\lambda_n(B_{X^*}\times(B(y_0^*;\delta)\cup B(-y_0^*;\delta)))\to1\qquad\text{as }n\to\infty,$$

and thus that

$$\lambda_n((B(x_0^*;\delta)\cup B(-x_0^*;\delta))\times(B(y_0^*;\delta)\cup B(-y_0^*;\delta)))\to1\qquad\text{as }n\to\infty. \tag{11}$$

Since, by the choice of the h_n^*'s, we also have $(x_n^{**}\otimes y_n^{**})(h_n^*)>\beta_n\beta_n'$ for all $n\in\mathbf{N}$, we conclude from Lemma 1.14 that $\|h_n^*-x_0^*\otimes y_0^*\|_{H^*}\to0$ as $n\to\infty$. This completes the proof of Theorem 3.2.

We close this paper with a further consequence of the results given so far.

3.4. Lemma. *Whenever* $\dim X \geq 2$ *and* $\dim Y \geq 2$, *and H is a linear subspace of* $K_{w^*}(X^*, Y)$, *containing* $X \otimes Y$, *then the set* $\operatorname{ext} B_{H^*}$ *is not norm-dense in* S_{H^*}.

Proof. Suppose that $\| \; \|\text{-cl}(\operatorname{ext} B_{H^*}) = S_{H^*}$. Then we have:

$$S_{H^*} = \| \; \|\text{-cl}(\operatorname{ext} B_{H^*}) = \| \; \|\text{-cl}(\operatorname{ext} B_{X^*} \otimes \operatorname{ext} B_{Y^*}).$$

Thus, if $T_0 \in S_{H^*}$, then there exists a sequence $((x_n^*, y_n^*))_n \subset \operatorname{ext} B_{X^*} \times \operatorname{ext} B_{Y^*}$ such that $\| T_0 - x_n^* \otimes y_n^* \|_{H^*} \to 0$. Lemma 1.2 of section 1 thus reveals that $T_0 = x_0^* \otimes y_0^*$ for some $(x_0^*, y_0^*) \in S_{X^*} \times S_{Y^*}$. We conclude that $S_{H^*} = S_{X^*} \otimes S_{Y^*}$, which contradicts the assumption that $\dim X \geq 2$ and $\dim Y \geq 2$. This completes the proof.

3.5. Corollary. *Whenever* $\dim X \geq 2$ *and* $\dim Y \geq 2$, *then neither* $X \tilde{\otimes}_\varepsilon Y$ *nor* $K(X, Y)$ *is smooth, i.e. their (usual operator) norms are* not *Gateaux-differentiable at all non-zero elements.*

This follows from Lemma 3.4 and an easy application of the Bishop–Phelps Theorem, showing that $\| \; \|\text{-cl}(\text{w}^*\text{-exp}\, B_{Z^*}) = S_{Z^*}$ whenever the norm of Z is G-differentiable at every $z \in Z \setminus \{0\}$ (Z any Banach space).

The fact that, under the assumptions of Corollary 3.5, there always exists an operator $k_0 \in K(X, Y) \setminus \{0\}$ at which $\| \; \|_{K(X,Y)}$ is not Gateaux-differentiable, has been proved with different techniques by D.R. Lewis [12].

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